IMC Selection Test, May 2026
Organized by
Bhaskaracharya Pratishthana, Pune
Solutions and Scheme of marking
DAY-1

Date: 09/05/2026 Max. Marks: 50
Time: 8.30 AM to 1.30 PM
N.B.: Each question carries 10 marks.

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(x_1, \ldots, x_n) = \frac{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}}{x_1^{b_1} + \cdots + x_n^{b_n}} \text{ for } (x_1, \ldots, x_n) \neq (0, \ldots, 0) \text{ and } f(0, \ldots, 0) = 0,$$

where a_i's are positive integers and b_i's are even positive integers. Determine precisely all the a_i's and b_i's for which f is discontinuous at 0.
Solution: Writing $b_1 \leq b_2 \leq \cdots \leq b_n$, we claim that f is discontinuous if and only if $\sum_{i=1}^n \frac{a_i}{b_i} \leq 1$.
Write $c = \sum_{i=1}^n \frac{a_i}{b_i}$ and $g = x_1^{b_1} + \cdots + x_n^{b_n}$. Then

$$f = \frac{1}{g^{1-c}} \frac{x_1^{a_1}}{g^{a_1/b_1}} \cdots \frac{x_n^{a_n}}{g^{a_n/b_n}} = g^{c-1} \prod_{i=1}^n \frac{x_i^{a_i}}{g^{a_i/b_i}}.$$

If $c \leq 1$, then for any $t > 0$,

$$f(t^{1/b_1}, \cdots, t^{1/b_n}) = \frac{t^c}{nt} = \frac{t^{c-1}}{n}.$$

Therefore, when $c \leq 1$, the above does not tend to $f(0, \cdots, 0) = 0$ which means f is discontinuous at 0.
Conversely, suppose $c > 1$. As

$$\left| \frac{x_i^{a_i}}{g^{a_i/b_i}} \right| \leq \left| \frac{x_i^{a_i}}{(x_i^{b_i})^{a_i/b_i}} \right| = 1,$$

we have

$$|f(x_1, \cdots, x_n)| \leq g^{c-1}.$$

The RHS above tends to 0 as $(x_1, \cdots, x_n) \to 0$ because $c > 1$. Hence, f is continuous at 0.

2. Let G be a group in which exactly two elements (different from the unit element) are commuting. Show that G is isomorphic to either $\mathbb{Z}_3$ or S_3.
Solution: Since G is a group in which exactly two elements (different from the unit element) are commuting, G has at least 3 elements. If all the elements of G have order at most 2 then G is abelian and contains at least 4 elements, hence it does not satisfy the hypothesis of the problem.

Therefore G contains elements of order at least 3. Let $a \in G$ with $o(a) \geqslant 3$. If $o(a) > 3$ then a, a^2, a^3 are distinct and commute with each other. Hence it contradicts the hypothesis of the problem. Hence, $o(a) = 3$. It follows that $a \neq a^2$ and $a \cdot a^2 = a^2 \cdot a$, so a and a^2 are the two elements that commute.

Let $H = \{e, a, a^2\}$. If $G = H$ then G is isomorphic to Z_3 and we are done. If $G \neq H$, then every element of $G \setminus H$ must have order 2. (If it has an element of order 3, then it will contradict the hypothesis.)

Let $b \in G \setminus H$. If $x \in G \setminus H$, then both b and x are elements of order 2. If $bx \in G \setminus H$, then $(bx)^2 = e = b^2x^2$. Hence, $bx = xb$, a contradiction. Hence, $bx \in H$ and $x \in bH$, as $o(b = 2.)$ Therefore every element of $G \setminus H$ is in bH and $bh \subset G \setminus H$. Hence, G contains exactly 6 elements. Further, G is a non-abelian group and hence G is isomorphic to S_3.

3. Let $a_0, b_0 \in [0,1], a_n = \displaystyle\int_0^1 \max\{x, b_{n-1}\}dx$ and $b_n = \displaystyle\int_0^1 \min\{x, a_{n-1}\}dx, n \geq 1$. Find the limit of the sequence $\{a_n + b_n\}$, if it exists.

Solution: Assume that $0 \leq a_0, b_0 \leq 1$. Then $a_1 = \displaystyle\int_0^{b_0} b_0 dx + \int_{b_0}^1 x dx = \frac{1+b_0^2}{2}$ and $b_1 = \displaystyle\int_0^{a_0} x dx + \int_{a_0}^1 a_0 dx = a_0 - \frac{a_0^2}{2}$. By induction, it can be shown that

(a) $0 \leq a_n, b_n \leq 1$ for each n,

(b) $a_{n+1} = \dfrac{1+b_n^2}{2}$ and $b_{n+1} = a_n - \dfrac{a_n^2}{2}$ for each n.

(c) Since the integrands lie in $[0,1]$, we have

$$0 \leq a_n \leq 1, \qquad 0 \leq b_n \leq 1.$$

Also,

$$a_n - b_n = \int_0^1 \big(\max\{x, b_{n-1}\} - \min\{x, a_{n-1}\}\big)\, dx \geq 0,$$

hence

$$b_n \leq a_n \quad \text{for all } n.$$

(d) Assume that $\lim a_n = a$ and $\lim b_n = b$. Then from the above recurrence relation, we get $a = \frac{1+b^2}{2}$ and $b = a - \frac{a^2}{2}$. Therefore, $b = 1 \pm a$.

(e) Since $\min(x, a_{n-1}) \leq x$, it is clear that $b_n \leq 1/2$ and $\max(x, b_{n-1}) \leq x$ we have $a_n \geq 1/2$ for every $n \geq 1$. That is, $b_n \leq 1/2 \leq a_n$ for all n, so that $b \leq 1/2 \leq a$.

Hence, $b = 1 + a$ is not possible. Thus $a + b = 1$.

Alternatively, define $T : [0,1] \times [0,1] \to \mathbb{R}^2$ by

$$T(a,b) = \left(\int_0^1 \max(x,b)\, dx,\ \int_0^1 \min(x,a)\, dx\right)$$

Define the sup norm on $\mathbb{R}^2$:

$$\|(a,b)\|_\infty = \max\{|a|, |b|\}.$$

Then

$$T_1(b) = \int_0^1 \max(x,b)\, dx = \int_0^b b\, dx + \int_b^1 x\, dx = \frac{1+b^2}{2},$$

$$T_2(a) = \int_0^1 \min(x, a)\, dx = a - \frac{a^2}{2}$$

Thus,

$$T(a, b) = \left(\frac{1 + b^2}{2},\ a - \frac{a^2}{2}\right).$$

Let $(a_1, b_1), (a_2, b_2) \in [0, 1]^2$.

Then

$$|T_1(b_1) - T_1(b_2)| = \left|\frac{b_1^2 - b_2^2}{2}\right| = \frac{1}{2}|b_1 - b_2||b_1 + b_2|.$$

Since $b_1, b_2 \in [0, 1]$, we have $|b_1 + b_2| \leq 2$. Hence,

$$|T_1(b_1) - T_1(b_2)| \leq |b_1 - b_2|.$$

Also,

$$|T_2(a_1) - T_2(a_2)| = \left|(a_1 - a_2) - \frac{a_1^2 - a_2^2}{2}\right|$$

$$= |a_1 - a_2| \left|1 - \frac{a_1 + a_2}{2}\right|.$$

Since $a_1, a_2 \in [0, 1]$, we have:

$$0 \leq \frac{a_1 + a_2}{2} \leq 1 \quad \Rightarrow \quad \left|1 - \frac{a_1 + a_2}{2}\right| \leq 1.$$

Hence,

$$|T_2(a_1) - T_2(a_2)| \leq |a_1 - a_2|.$$

Thus,

$$\|T(a_1, b_1) - T(a_2, b_2)\|_\infty = \max\{|T_1(b_1) - T_1(b_2)|,\ |T_2(a_1) - T_2(a_2)|\}$$

$$\leq \max\{|b_1 - b_2|,\ |a_1 - a_2|\} = \|(a_1, b_1) - (a_2, b_2)\|_\infty$$

Further,

$$T_1'(b) = b \leq 1, \quad T_2'(a) = 1 - a \leq 1$$

In fact, on $[0, 1]$, both derivatives are strictly less than 1 except at endpoints, so there exists $k < 1$ such that:

$$\|T(a_1, b_1) - T(a_2, b_2)\|_\infty \leq k\|(a_1, b_1) - (a_2, b_2)\|_\infty$$

Hence T is a contraction on $[0, 1]^2$.

By Banach's contraction principle, there is unique $(a, b) \in [0, 1]^2$ such that

$$(a, b) = T(a, b)$$

$$a = \frac{1+b^2}{2}, \qquad b = a - \frac{a^2}{2}$$

Substitute $b = a - \frac{a^2}{2}$ into the first equation:

$$a = \frac{1 + \left(a - \frac{a^2}{2}\right)^2}{2}$$

Solving, we obtain

$$a = 2 - \sqrt{2}, \qquad b = \sqrt{2} - 1$$

The unique fixed point is

$$\boxed{(a, b) = (2 - \sqrt{2},\ \sqrt{2} - 1)}$$

Also,

$$(2 - \sqrt{2}) + (\sqrt{2} - 1) = 1.$$

Let $a_0, b_0 \in [0, 1]$. Then $T(a_n, b_n) = (a_{n+1}, b_{n+1})$, so that $\lim\limits_{n\to\infty} T(a_n, b_n) = \lim\limits_{n\to\infty} (a_{n+1}, b_{n+1})$. Therefore, $T(\lim\limits_{n\to\infty} a_n, \lim\limits_{n\to\infty} b_n) = (\lim\limits_{n\to\infty} a_{n+1}, \lim\limits_{n\to\infty} b_{n+1}) \Rightarrow T(a, b) = (a, b)$. Thus, (a, b) is the unique fixed point of T, where $a = \lim\limits_{n\to\infty} a_n, b = \lim\limits_{n\to\infty} b_n$.

4. Define a function $f : \mathbb{R} \mapsto \mathbb{R}$ as follows:

   (a) $f(p) =$ product of greatest and least roots of $q(x) = x^3 - 3x - p$, if $q(x)$ has three real roots.

   (b) $f(p) =$ the square of the real root if $q(x)$ has only one real root.

   Determine the minimum value of $f(p)$ as p varies over all real numbers.
   Solution:
   (a) Let $x^3 - 3x - p = (x - a)(x - b)(x - c)$ and $a \le b \le c$. Therefore, $f(p) = ac$. We have, $a + b + c = 0, ab + bc + ac = -3$ and $abc = p$. Since $a + c = -b$, we get $-3 = ab + bc + ac = b(a + c) + ac = -b^2 + ac$. Therefore, $f(p) = ac = b^2 - 3$. Hence $f(p) \ge -3$ and $f(p) = -3$ if $b = 0$. But if $b = 0$, then $p = abc = 0$. Then we get, $x^3 - 3x = 0$, which gives $x = 0, \pm\sqrt{3}$. Thus, $f(p) = -3$.
   (b) Let $x^3 - 3x - p = (x - a)(x^2 + qx + r)$, where $q^2 - 4r < 0$. We have $-a + q = 0$ and $r - aq = -3$. Now, $f(p) = a^2 = aq = r + 3$. Since $q^2 - 4r < 0$, we get $r > 0$. Therefore, $f(p) > 3$. Hence the minimum value of $f(p) = -3$.

5. What are all possible ways to fill in the blanks with single digits such that the following sentence is true?
   "In this sentence, the number of occurrences of the digit 0 is ____, of the digit 1 is ____, of the digit 2 is ____, of the digit 3 is ____, of even numbers is ____, of odd numbers is ____, and of prime numbers is____."
   Solution: We show that the only two solutions are $(1, 2, 3, 2, 6, 5, 6)$ and $(1, 2, 3, 2, 5, 6, 7)$. Let us call the digits to be filled in the blanks as $(a_0, a_1, a_2, a_3, a_{ev}, a_{odd}, a_{prime})$. From the four built-in digits, we have $a_0, a_1, a_2, a_3 \ge 1$ and the other three are all ≥ 2. Therefore, $a_0 = 1, a_1 \ge 2$ and $a_{odd} \ge 3$. Since $a_{ev} + a_{odd} = 11$, they have the opposite parity. Thus, there is at least one more each of odd and even

entries. That is, $a_{ev} \geq 3$ and $a_{odd} \geq 4$. Now the five possibilities for (a_{ev}, a_{odd}) are $(3,8), (4,7), (5,6), (6,5)$ and $(7,4)$. In each of these pairs, exactly one of them is a prime number. Therefore, $a_{prime} \geq 3$. If $a_{prime} = 3$, this would be the fourth prime in the statement; hence $a_{prime} > 3$.
Now a_2 and a_3 are the only remaining places where the digit 1 can occur. Thus $a_1 \leq 4$. If $a_1 = 4$ then there are three remaining primes, counting the one from (a_{ev}, a_{odd}) but not the value of a_{prime}. But then $a_{prime} = 4$ would give a contradiction as there would be only three primes, and any other value of a_{prime} would be too large.
Thus, we have $a_1 \leq 3$. Suppose $a_1 = 3$; we will derive a contradiction.
If $a_1 = 3$, then $a_3 \geq 2$. We claim $a_3 \geq 3$ because if $a_3 = 2$, then $a_2 \geq 2$ and we cannot have $a_1 = 3$. We claim that $a_3 = 3$; else $a_3 \geq 4$ and we would have $a_{prime} = 3$ which is not true as we will have more than three primes then. Thus, $a_3 = 3$ and we would have five primes, counting the one from (a_{ev}, a_{odd}) but not the value of a_{prime}. But, if $a_{prime} = 5$, we would have six primes, and if $a_{prime} = 6$, we would have five primes. This is a contradiction from which we deduce that $a_1 < 3$ and hence $a_1 = 2$.
So $a_2 \geq 3$. If $a_2 = 3$, we would have three "2"'s which is a contradiction showing that $a_2 \geq 3$. Since only $a_3 = 2$ allows us to have a third "2", we must have both $a_3 = 2$ and $a_2 = 3$. Now there are six primes counting the one from (a_{ev}, a_{odd}) but not the value of a_{prime}. Therefore, $a_{prime} = 6$ or 7 and we get the two solutions

$$(a_0, a_1, a_2, a_3, a_{ev}, a_{odd}, a_{prime}) = (1,2,3,2,6,5,6)\ ,\ (1,2,3,2,5,6,7).$$

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Solutions and Scheme of marking
DAY-2

Date: 10/05/2026 Max. Marks: 50
Time: 8.30 AM to 1.30 PM
N.B.: Each question carries 10 marks.

1. Let $\{x_n\}$ be a sequence of real numbers defined by $x_1 = 1$ and

$$x_{n+1} = \frac{n^2}{x_n} + \frac{x_n}{n^2} + 2$$

for all $n \geq 1$. Prove that $\lim_{n\to\infty} (x_n - n) = \dfrac{1}{2}$.
Solution: Define

$$a_n = \frac{n^2}{n - 1/2}, \quad b_n = \frac{(n-1/2)^2}{n - 3/2}$$

for all $n \geq 1$.
We show by induction that $a_n \leq x_n \leq b_n$ for all $n \geq 3$.
Observe that $a_3 = 18/3 < x_3 = 4 < 25/6 = b_3$. Suppose $a_n \leq x_n \leq b_n$.
Consider the functions

$$f_n(y) = \frac{n^2}{y} + \frac{y}{n^2} + 2.$$

Observe that

$$f_n'(y) = \frac{y^2 - n^4}{n^2 y^2}.$$

Note that $f_n(y)$ is decreasing on the interval $(0, n^2)$ and $a_n, b_n \in (0, n^2)$ for $n \geq 3$.
Thus, by induction
$f_n(b_n) \leq x_{n+1} = f_n(x_n) \leq f_n(a_n)$.
It remains to show that $a_{n+1} \leq f_n(b_n)$ and $f_n(a_n) \leq b_{n+1}$.
Note that $f_n(a_n) = \dfrac{n^2}{a_n} + \dfrac{a_n}{n^2} + 2 = (n - 1/2) + \dfrac{1}{(n-1/2)} + 2 = b_{n+1}.$
Note that $f_n(b_n) - a_{n+1} = \dfrac{n^2}{b_n} + \dfrac{b_n}{n^2} + 2 - \dfrac{(n+1)^2}{n + 1/2}$
$= \dfrac{16n^3 + 8n^2 - 6n + 1}{2n^2(2n-3)(2n-1)^2(2n+1)} > 0$ for all $n \geq 3$.
Therefore $a_{n+1} \leq x_{n+1} \leq b_{n+1}$. Hence $a_n \leq x_n \leq b_n$ for all $n \geq 3$.
We now show that $\lim\limits_{n\to\infty} (a_n - n) = \lim\limits_{n\to\infty} (b_n - n) = \dfrac{1}{2}$.
Observe that $a_n - n = \dfrac{n}{2n-1}$ and $b_n - n = \dfrac{n/2 - 1/4}{n - 3/2}$.
Hence by Sandwich principle,

$$\lim_{n\to\infty} (a_n - n) = \lim_{n\to\infty} (x_n - n) = \lim_{n\to\infty} (b_n - n) = \frac{1}{2}.$$

2. Anand is going to play two games of chess with an opponent whom he has never played against before. The opponent is equally likely to be a beginner, intermediate, or a master. Depending on which, Anand's chances of winning an individual game are 90%, 50% or 30%, respectively. Assume that, given the skill level of the opponent, the outcomes of the games are independent. Given that Anand wins the first game, what is the probability that he will also win the second game?
Solution: Let W_1 be the event of Anand winning the first game. By the law of total probability, $\text{Prob}(W_1) = \dfrac{0.9 + 0.5 + 0.3}{3} = \dfrac{17}{30}$.

Let W_2 be the event of Anand winning the second game. Thus we have $\text{Prob}((W_2|W_1) = \dfrac{\text{Prob}(W_2 \cap W_1)}{\text{Prob}(W_1)}$.
The Law of total probability gives $\text{Prob}(W_2 \cap W_1)$ by

$$= \frac{1}{3}\left(\text{Prob}(W_2 \cap W_1 \mid \text{beginner}) + \text{Prob}(W_2 \cap W_1 \mid \text{intermediate}) + \text{Prob}(W_2 \cap W_1 \mid \text{master})\right).$$

Since W_1 and W_2 are conditionally independent given the skill level of the opponent, the above equals $\dfrac{1}{3}(\,(0.9)^2 + (0.5)^2 + (0.3)^2)) = \dfrac{23}{60}$. This gives the required probability $= \dfrac{23}{60} \times \dfrac{30}{17} = \dfrac{23}{34}$.

3. Let us write $\mathbb{Z}_{12} = \{0, 1, \ldots, 11\}$. A triad (a, b, c) is a three-tuple of distinct elements of $\{0, 1, \ldots, 11\}$. Two triads (a, b, c) and (a', b', c') are defined to be equivalent either if $(a, b, c) = (a', b', c')$ or if there is $k \in \mathbb{Z}$ such that $a' \equiv (a + k) \pmod{12}$, $b' \equiv (b + k) \pmod{12}$, $c' \equiv (c + k) \pmod{12}$. Find the number of

equivalence classes of triads.
Solution: The equivalence class of a triad $(a,b,c) = \{(a+k \bmod 12, b+k \bmod 12, c+k \bmod 12) : 0 \leq k \leq 11\}$. All these elements $(a+k \bmod 12, b+k \bmod 12, c+k \bmod 12)$ are distinct and hence the size of each equivalence class is 12. Since there are $12 \times 11 \times 10 = 1320$ triads, the number of equivalence classes is $1320/12 = 110$.

4. Let $f(x)$ be a monic polynomial of degree n with distinct real zeros, say $a_1 < a_2 < \cdots < a_n$. Let $\delta(f) = \min(a_i - a_{i-1})$. Show that $\delta(f') > \delta(f)$.
Solution: We have $f(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$. Then

$$\frac{f'(x)}{f(x)} = \sum_{k=1}^{n} \frac{1}{(x-a_k)}$$

Let $g(x) = \dfrac{f'(x)}{f(x)}$. Suppose y_0 and y_1 are real numbers such that, for some j,

$$y_0 < a_j < y_1 \leq y_0 + \delta(f)$$

Then we have $y_1 - y_0 \leq \delta(f) \leq a_{i+1} - a_i \ \forall i$. Hence for $1 \leq i \leq j-1$, we have $y_0 - a_i \geq y_1 - a_{i+1} > 0$ and so $\dfrac{1}{y_0 - a_i} \leq \dfrac{1}{y_1 - a_{i+1}}$. Similarly, for $j \leq i \leq n-1$, we have $0 > y_0 - a_i \geq y_1 - a_{i+1}$ and again $\dfrac{1}{y_0 - a_i} \leq \dfrac{1}{y_1 - a_{i+1}}$.
Finally, $y_0 - a_n < 0 < y_1 - a_1$, so $\dfrac{1}{y_0 - a_n} < 0 < \dfrac{1}{y_1 - a_1}$.
Adding these inequalities we get that $g(y_0) < g(y_1)$.
Now suppose y_0, y_1 are the roots of f' with $y_0 < y_1$. Then $g(y_0) = g(y_1) = 0$ and there exists j such that $y_0 < x_j < y_1$. So we cannot have $y_1 \leq y_0 + \delta(f)$. Thus $y_1 - y_0 > \delta(f)$ implying that $\delta(f') > \delta(f)$.

5. (a) Let A, B be 2×2 matrices with entries complex numbers. If $A^2 + B^2 = 2AB$, prove that $AB = BA$ and Trace (A) =Trace (B).

   (b) If the entries of A and B are from $\mathbb{Z}_2$, and if $A^2 + B^2 = 2AB$, is it true that $AB = BA$? Justify.

   (c) If A, B are 3×3 matrices with entries complex numbers satisfying $A^3 - B^3 = 3(A^2B - AB^2)$, is it true that $AB = BA$? Justify.

   Solution:

   (a) We first show that A and B have same eigenvalues. Suppose λ is a an eigenvalue of B with associated eigenvector $v \in \mathbb{C}^n$. Thus we have $Bv = \lambda v$ and $B^2 v = \lambda^2 v$. So the given equation $A^2 - 2AB + B^2 = 0$ implies

$$0 = (A^2 - 2AB + B^2)v = (A^2 - 2\lambda Av + \lambda^2 v) = (A - \lambda I)^2)v$$

   Thus λ is an eigenvalue of A.
   By taking the transpose of the given equation, it follows that an eigenvalue of A is also an eigenvalue of B. As the matrix is of the size 2×2, the multiplicities of the eigenvalues also coincide (one eigenvalue of multiplicity 2 or two eigenvalues, each of multiplicity 1). Thus $Trace(A) = Trace(B)$. (in fact the characteristic polynomials of A and B are equal).

Now by the given equation, we have $(A-B)^2 = AB - BA$. Thus $Trace(A-B)^2 = 0$. We also have $Trace(A-B) = 0$. Thus we get that $(A-B)$ is nilpotent and hence $AB - BA = (A-B)^2 = 0$.

(b) If the entries are in $\mathbb{Z}_2$, it is not true that the given equation implies the commutativity of A and B. For example, let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Then $AB = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $BA = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and

$A^2 + B^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(c) No. The conclusion does not hold under the given assumption. Choose A, B as follows:

$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$.